

ON CONCENTRATION OF LEAST ENERGY SOLUTIONS FOR MAGNETIC CRITICAL CHOQUARD EQUATIONS

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Abstract

In the present paper, we consider the following magnetic nonlinear Choquard equation

$$\begin{cases} (-i\nabla + A(x))^2 u + \mu g(x)u = \lambda u + (|x|^{-\alpha} * |u|^{2_\alpha^*})|u|^{2_\alpha^*-2}u, & u > 0 \text{ in } \mathbb{R}^n, \\ u \in H^1(\mathbb{R}^n, \mathbb{C}) \end{cases}$$

where $n \geq 4$, $2_\alpha^* = \frac{2n-\alpha}{n-2}$, $\lambda > 0$, $\mu \in \mathbb{R}$ is a parameter, $\alpha \in (0, n)$, $A(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a magnetic vector potential and $g(x)$ is a real valued potential function on \mathbb{R}^n . Using variational methods, we establish the existence of least energy solution under some suitable conditions. Moreover, the concentration behavior of solutions is also studied as $\mu \rightarrow +\infty$.

Key words: Nonlinear Schrödinger equations, Magnetic potential, Choquard equation, Critical exponent.

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1 Introduction and main results

In this article, we study the existence and concentration behavior of nontrivial solutions of the following nonlinear Schrödinger equation with nonlocal nonlinearity

$$(P_{\lambda, \mu}) \begin{cases} (-i\nabla + A(x))^2 u + \mu g(x)u = \lambda u + (|x|^{-\alpha} * |u|^{2_\alpha^*})|u|^{2_\alpha^*-2}u, & u > 0 \text{ in } \mathbb{R}^n \\ u \in H^1(\mathbb{R}^n, \mathbb{C}) \end{cases}$$

where $n \geq 4$, $2_\alpha^* = \frac{2n-\alpha}{n-2}$, $\mu > 0$, $\lambda \in \mathbb{R}$, $A = (A_1, A_2, \dots, A_n) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a vector (or magnetic) potential such that $A \in L_{\text{loc}}^2(\mathbb{R}^n, \mathbb{R}^n)$ and $g(x)$ satisfies the following assumptions:

- (g1) $g \in C(\mathbb{R}^n, \mathbb{R})$, $g \geq 0$ and $\Omega := \text{interior of } g^{-1}(0)$ is a nonempty bounded set with smooth boundary and $\overline{\Omega} = g^{-1}(0)$.
- (g2) There exists $M > 0$ such that $\text{meas}\{x \in \mathbb{R}^n : g(x) \leq M\} < \infty$, where meas denotes the Lebesgue measure in \mathbb{R}^n .

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The more general form of the above problem

$$(-i\nabla + A(x))^2 w + G(x)w = F(x, w), \quad w \in H^1(\mathbb{R}^n, \mathbb{C}) \quad (1.1)$$

arises when we try to look for standing wave solution of the Schrödinger equation

$$i\hbar \frac{\partial \phi}{\partial t} = (-i\hbar \nabla + A(x))^2 \phi + Q(x)\phi - n(x, |\phi|)\phi,$$

where \hbar is the Plank constant. A lot of attention has been paid to nonlinear Schrödinger equation in recent years. When $A \equiv 0$ (no magnetic potential) in (1.1), many authors studied the problem, refer [5, 7, 12]. The problem of the type

$$-\Delta u + \mu a(x)u = \lambda u + |u|^{p-2}u, \quad (1.2)$$

where $a \geq 0$ is potential well, with subcritical growth i.e. $p < 2^* = 2n/(n-2)$ has been investigated extensively in [4, 8, 9, 25, 29]. In critical case $p = 2^*$, Clapp and Ding [14] established existence and multiplicity of positive solutions of the problem using variational methods. For Schrödinger equations with critical nonlinearity, we refer to [16, 11, 2, 27].

When the magnetic vector potential $A \not\equiv 0$, the Schrödinger equation of the form

$$(-i\hbar \nabla + A(x))^2 u + V(x)u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^n,$$

where V is electric potential function, has been widely studied by many authors, refer [10, 13, 21] and the references therein. Motivated by these results, very recently Lü [24] studied the problem

$$(-i\nabla + A(x))^2 u + (g_0(x) + \mu g(x))u = (|x|^{-\alpha} * |u|^p)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^n, \mathbb{C}), \quad (1.3)$$

where $n \geq 3$, $\alpha \in (0, n)$, $p \in \left(\frac{2n-\alpha}{n}, \frac{2n-\alpha}{n-2}\right)$, g_0 and g are real valued functions on \mathbb{R}^n satisfying some conditions and $\mu > 0$. He proved the existence of ground state solution for $\mu \geq \mu^*$, for some $\mu^* > 0$ and concentration behavior of solutions as $\mu \rightarrow \infty$. The Hardy-Littlewood-Sobolev inequality (see Theorem 2.2) plays an important role for studying such problems and in that context, we say $2_\alpha^* = \frac{2n-\alpha}{n-2}$ is the critical exponent. When $A \equiv 0$, $g_0 \equiv 0$, $g \equiv 1$ and $\mu = 1$ in (1.3), that is

$$-\Delta u + u = (|x|^{-\alpha} * |u|^p)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^n)$$

are generally called the Choquard equation which arise in various fields of physics, example quantum theory of large systems of nonrelativistic bosonic atoms and molecules. Choquard equations are another topic of attraction for researchers now a days which in turn rendered a huge literature in this area [26, 15, 20]. In [22], Lieb proved the existence and uniqueness, up to translations, of the ground state for the problem

$$-\Delta u + u = (|x|^\mu * F(u))f(u) \quad \text{in } \mathbb{R}^n,$$

where $f(t)$ is critical growth nonlinearity such that $|tf(t)| \leq C||t|^2 + |t|^{\frac{2n-\mu}{n-2}}|$ for $t \in \mathbb{R}$, $\mu > 0$, for some $C > 0$ and $F(t) = \int_0^t f(z)dz$. In [3, 18, 19], Gao and Yang showed existence and multiplicity results for Brezis-Nirenberg type problem of the nonlinear Choquard equation

$$-\Delta u = \left(\int_{\Omega} \frac{|u|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dy \right) |u|^{2_{\alpha}^*-2} u + \lambda g(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is smooth bounded domain in \mathbb{R}^n , $n > 2$, $\lambda > 0$, $0 < \alpha < n$ and $g(u)$ is a nonlinearity with certain assumptions.

Now a very obvious question arises, what happens in the critical case $p = 2_{\alpha}^*$ in (1.3). Here in this paper, we consider the problem $(P_{\lambda,\mu})$ which is motivated by (1.2) and (1.3). The main difficulty for this problem is the presence of critical power nonlinearity in the sense of Hardy-Littlewood-Sobolev inequality which is also nonlocal in nature. The critical exponent term being nonlocal adds on the difficulty to study the Palais-Smale level around a nontrivial critical point. We define $\nabla_A u = (-i\nabla + A(x))u$ and consider the minimization problem here by defining

$$S_A = \inf_{u \in H_A^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu g(x)|u|^2) dx}{\int_{\mathbb{R}^n} (|x|^{\alpha} * |u|^{2_{\alpha}^*}) |u|^{2_{\alpha}^*} dx}$$

and proved that it is attained under some necessary and sufficient conditions which is a new result. Also the other results proved here are new and there is no work concerning this problem to the best of our knowledge. Following the approach of [14], we show that $(P_{\lambda,\mu})$ has a solution. Also we show that the problem (P_{λ})

$$(P_{\lambda}) \begin{cases} (-i\nabla + A(x))^2 u = \lambda u + (|x|^{\alpha} * |u|^{2_{\alpha}^*}) u^{2_{\alpha}^*-1}, & u > 0 \text{ in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases},$$

for small λ acts as a limit problem for $(P_{\lambda,\mu})$ as $\mu \rightarrow \infty$. We use the knowledge of (P_{λ}) to show the concentration behavior of solutions of $(P_{\lambda,\mu})$. Before stating our main results, we give some definitions.

Definition 1.1 A solution u_{μ} is said to be a least energy solution if the energy functional

$$I_{\lambda,\mu}(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2} (|\nabla_A u|^2 + (\mu g(x) - \lambda)|u|^2) - \frac{1}{22_{\alpha}^*} (|x|^{-\alpha} * |u|^{2_{\alpha}^*}) |u|^{2_{\alpha}^*} \right) dx$$

achieves its minimum at u_{μ} over all nontrivial solutions of $(P_{\lambda,\mu})$.

Definition 1.2 A sequence of solutions $\{u_k\}$ of (P_{λ,μ_k}) is said to concentrate at a solution u of (P_{λ}) if a subsequence converges strongly to u in $H_A^1(\mathbb{R}^n)$ as $\mu_k \rightarrow \infty$, where $H_A^1(\mathbb{R}^n)$ is defined in section 2.

Our main result includes the following theorems.

Theorem 1.3 Under the assumptions (g1) and (g2), for every $\lambda \in (0, \lambda_1(\Omega))$ there exists $\mu(\lambda) > 0$ such that $(P_{\lambda,\mu})$ has a least energy solution u_{μ} for each $\mu \geq \mu(\lambda)$.

Theorem 1.4 Let $\{u_m\}$ be a sequence of non-trivial solutions of (P_{λ,μ_m}) with $\mu_m \rightarrow \infty$ and $I_{\lambda,\mu_m}(u_m) \rightarrow c < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}$ as $m \rightarrow \infty$. Then u_{μ_m} concentrates at a solution of (P_{λ}) .

2 Variational Setting and Preliminaries

We assume that (g_1) and (g_2) are satisfied throughout this paper. Let us define

$$H_A^1(\mathbb{R}^n, \mathbb{C}) = \{u \in L^2(\mathbb{R}^n, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^n, \mathbb{C}^n)\}.$$

Then $H_A^1(\mathbb{R}^n, \mathbb{C})$ is a Hilbert space with the inner product

$$\langle u, v \rangle_A = \operatorname{Re} \left(\int_{\mathbb{R}^n} \nabla_A u \overline{\nabla_A v} \, dx \right),$$

where $\operatorname{Re}(w)$ denotes the real part of $w \in \mathbb{C}$ and \bar{w} denotes its complex conjugate. The associated norm $\|\cdot\|_A$ on the space $H_A^1(\mathbb{R}^n, \mathbb{C})$ is given by

$$\|u\|_A = \left(\int_{\mathbb{R}^n} |\nabla_A u|^2 \, dx \right)^{\frac{1}{2}}.$$

We call $H_A^1(\mathbb{R}^n, \mathbb{C})$ simply $H_A^1(\mathbb{R}^n)$ and $H_A^1(\Omega)$ is defined similarly. Let $H_A^{0,1}(\Omega, \mathbb{C})$ (denoted by $H_A^{0,1}(\Omega)$ for simplicity) be the Hilbert space defined by the closure of $C_c^\infty(\Omega, \mathbb{C})$ under the scalar product $\langle u, v \rangle_A$, where $\Omega = \text{interior of } g^{-1}(0)$. Thus

$$\|u\|_{H_A^{0,1}(\Omega)} = \left(\int_{\Omega} |\nabla_A u|^2 \, dx \right)^{\frac{1}{2}}.$$

Let $E = \{u \in H_A^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} g(x)|u|^2 \, dx < +\infty\}$ be the Hilbert space equipped with the inner product

$$\langle u, v \rangle = \operatorname{Re} \left(\int_{\mathbb{R}^n} (\nabla_A u \overline{\nabla_A v} \, dx + g(x)u\bar{v}) \, dx \right)$$

and the associated norm $\|\cdot\|_E$, where

$$\|u\|_E^2 = \int_{\mathbb{R}^n} (|\nabla_A u|^2 + g(x)|u|^2) \, dx.$$

Then $\|\cdot\|_E$ is clearly equivalent to each of the norm $\|\cdot\|_\mu$, where

$$\|u\|_\mu^2 = \int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu g(x)|u|^2) \, dx$$

for $\mu > 0$. We have the following well known *diamagnetic inequality* (for proof, see [23], Theorem 7.21).

Theorem 2.1 *If $u \in H_A^1(\mathbb{R}^n)$, then $|u| \in H^1(\mathbb{R}^n)$ and*

$$\nabla|u|(x) \leq |\nabla u(x) + iA(x)u(x)| \text{ for a.e. } x \in \mathbb{R}^n.$$

This theorem says that if $u \in H_A^1(\mathbb{R}^n, \mathbb{C})$ then $|u| \in H^1(\mathbb{R}^n, \mathbb{R})$. This implies $H_A^1(\Omega) \hookrightarrow L^q(\mathbb{R}^n, \mathbb{C})$ is continuous for each $1 \leq q \leq 2^*$ and compact $1 \leq q < 2^*$, where $2^* = \frac{2n}{n-2}$ is the

Sobolev critical exponent. So, for each $q \in [2, 2^*]$, there exists constant $b_q > 0$ (independent of μ) such that

$$|u|_q \leq b_q \|u\|_\mu, \text{ for any } u \in E, \quad (2.1)$$

where $|\cdot|$ denotes the norm in $L^q(\mathbb{R}^n, \mathbb{C})$. Let us denote

$$B(u) = \int_{\mathbb{R}^n} (|x|^\alpha * |u|^{2_\alpha^*}) |u|^{2_\alpha^*} dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^{2_\alpha^*} |u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy. \quad (2.2)$$

To estimate the nonlocal term $B(u)$, we have the following Hardy-Littlewood-Sobolev inequality (refer [23], Theorem 4.3).

Proposition 2.2 *Let $t, r > 1$ and $0 < \alpha < n$ with $1/t + \mu/n + 1/r = 2$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. There exists a sharp constant $C(t, n, \alpha, r)$, independent of f, h such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\alpha} dx dy \leq C(t, n, \alpha, r) |f|_t |h|_r.$$

Proposition 2.2 implies that

$$|B(u)| \leq C(\alpha, n, 2_\alpha^*) |u|_{2_\alpha^*}^{22_\alpha^*}, \quad (2.3)$$

where $C(\alpha, n, 2_\alpha^*)$ is a positive constant. By (2.1), we say that $B(u)$ is well defined for $u \in E$. Also $B(u) \in C^1(E, \mathbb{R})$.

Definition 2.3 *We say that a function $u \in H_A^1(\mathbb{R}^n)$ is a weak solution of $(P_{\lambda, \mu})$ if*

$$\operatorname{Re} \left(\int_{\mathbb{R}^n} \nabla_A u \overline{\nabla_A v} dx + \int_{\mathbb{R}^n} (\mu g(x) - \lambda) u \bar{v} dx - \int_{\mathbb{R}^n} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^*-2} u \bar{v} dx \right) = 0$$

for all $v \in H_A^1(\mathbb{R}^n)$.

The main idea to prove the existence of solution for the problem $(P_{\lambda, \mu})$ is using variational methods where the weak solutions for $(P_{\lambda, \mu})$ are obtained by finding the critical points of the energy functional $I_{\lambda, \mu} : H_A^1(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined by

$$I_{\lambda, \mu}(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2} (|\nabla_A u|^2 + (\mu g(x) - \lambda) |u|^2) - \frac{1}{22_\alpha^*} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^*} \right) dx.$$

Then $I_{\lambda, \mu} \in C^1(H_A^1(\mathbb{R}^n), \mathbb{R})$ with

$$\langle I'_{\lambda, \mu}(u), v \rangle = \operatorname{Re} \left(\int_{\mathbb{R}^n} \nabla_A u \overline{\nabla_A v} dx + \int_{\mathbb{R}^n} (\mu g(x) - \lambda - (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^*-2}) u \bar{v} dx \right)$$

for $u, v \in H_A^1(\mathbb{R}^n)$. Thus we characterize the weak solutions of $(P_{\lambda, \mu})$ as the critical points of $I_{\lambda, \mu}$. From now onwards, we denote $\lambda_1 > 0$ as the best constant of the compact embedding $H_A^1(\Omega) \hookrightarrow L^2(\Omega, \mathbb{C})$ given by

$$\lambda_1 = \inf_{u \in H_A^1(\Omega)} \left\{ \int_{\Omega} |\nabla_A u|^2 dx : \int_{\Omega} |u|^2 dx = 1 \right\}$$

which is also the first eigenvalue of $-\Delta_A := (-i\nabla + A)^2$ on Ω with boundary condition $u = 0$. Let S denote the best Sobolev constant of the embedding $H_0^1(\Omega, \mathbb{R}) \hookrightarrow L^{2^*}(\Omega, \mathbb{R})$ which is given by

$$S = \inf_{u \in H_0^1(\Omega, \mathbb{R})} \left\{ \int_{\Omega} |\nabla u|^2 \, dx : \int_{\Omega} |u|^{2^*} \, dx = 1 \right\}.$$

We know that S is independent of Ω and it is achieved if only if $\Omega = \mathbb{R}^n$. We use $S_{H,L}$ to denote the best constant for (2.3) as

$$S_{H,L} = \inf_{u \in H^1(\mathbb{R}^n, \mathbb{R})} \left\{ \int_{\Omega} |\nabla u|^2 \, dx : B(u) = 1 \right\}.$$

By Lemma 1.2 of [18], we get that $S_{H,L}$ is achieved by the family of functions

$$U(x) = \frac{C_n}{(b^2 + |x - a|^2)^{(n-2)/2}}$$

where $C_n, b > 0$ and $a \in \mathbb{R}$ are constants.

3 Palais Smale analysis and compactness results

In this section, we find the Palais critical threshold below which any Palais Smale $(PS)_c$ sequence has a convergent subsequence. We recall that a sequence $\{u_m\} \subset E$ is said to be a $(PS)_c$ sequence (for $I_{\lambda,\mu}$) if $I_{\lambda,\mu}(u_m) \rightarrow c$ and $I'_{\lambda,\mu}(u_m) \rightarrow 0$ as $m \rightarrow \infty$. We say that $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition if every $(PS)_c$ sequence contains a convergent subsequence.

Lemma 3.1 *Suppose $\mu_m \geq 1$ and $u_m \in E$ be such that $\mu_m \rightarrow \infty$ as $m \rightarrow \infty$ and $\|u_m\|_{\mu_m} < K$, for all $m \in \mathbb{N}$. Then there exist $u \in H_A^{0,1}(\Omega)$ such that (upto a subsequence), $u_m \rightharpoonup u$ weakly in E as $m \rightarrow \infty$ and $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$.*

Proof. Since the norms $\|\cdot\|_E$ and $\|\cdot\|_{\mu}$ are equivalent, we have $\|u_m\|_E^2 < K'$, for some constant $K' > 0$. So there exists $u \in E$ such that $u_m \rightharpoonup u$ weakly in E and $u_m \rightarrow u$ strongly in $L_{\text{loc}}^2(\mathbb{R}^n)$ as $m \rightarrow \infty$. Let $C_r = \{x : |x| \leq r, g(x) \geq 1/r\}$, $r \in \mathbb{N}$. Then we can easily see that

$$\int_{C_r} |u_m|^2 \, dx \leq r \int_{C_r} g(x) |u_m|^2 \, dx \leq \frac{r}{\mu_m} \|u_m\|_{\mu_m}^2 \leq \frac{rk'}{\mu_m} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

This holds for every r which implies $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Since $\partial\Omega$ is smooth, we have $u \in H_A^{0,1}(\Omega)$. The next step is to show $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$. Let $D = \{x \in \mathbb{R}^n : g(x) \leq M\}$, where M is defined as in (g2). Then

$$\int_{\mathbb{R}^n \setminus D} u_m^2 \, dx \leq \frac{1}{\mu_m M} \int_{\mathbb{R}^n \setminus D} \mu_m g(x) u_m^2 \, dx \leq \frac{K}{\mu_m M} \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.1)$$

Let $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$ and $q \in \left(1, \frac{n}{n-2}\right)$ such that $q' = \frac{q}{q-1}$. Then using (2.1) and equivalence of norms, we have

$$\int_{B_R^c \cap D} |u_m - u|^2 \, dx \leq |u_m - u|_{2q}^2 (\text{meas}(B_R^c \cap D))^{1/q'} \leq C_1 b_{2q}^2 \|u_m - u\|_E^2 (\text{meas}(B_R^c \cap D))^{1/q'},$$

where C_1 is a positive constant and $B_R^c = \mathbb{R}^n \setminus B_R$. Hence by (g2) we get

$$\int_{B_R^c \cap D} |u_m - u|^2 \, dx \rightarrow 0 \text{ as } R \rightarrow \infty. \quad (3.2)$$

Lastly, as we know $u_m \rightarrow u$ strongly in $L_{\text{loc}}^2(\mathbb{R}^n)$ we get

$$\int_{B_R} |u_m - u|^2 \, dx \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.3)$$

Therefore using (3.1), (3.2) and (3.3), we get $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$ as $m \rightarrow \infty$. \blacksquare

Now let $T_\mu := -\Delta_A + \mu g(x)$ be an operator defined on E as

$$(T_\mu u, v) = \operatorname{Re} \left(\int_{\mathbb{R}^n} (\nabla_A u \overline{\nabla_A v} + \mu g(x) u \bar{v}) \, dx \right).$$

Clearly T_μ is a self adjoint operator and if $a_\mu := \inf \sigma(T_\mu)$, i.e. the infimum of the spectrum of T_μ . Then a_μ can be characterized as

$$0 \leq a_\mu = \inf \{ (T_\mu(u), u) : u \in E, |u|_2 = 1 \} = \inf \{ \|u\|_\mu^2 : u \in E, |u|_2 = 1 \}$$

and thus, a_μ is nondecreasing in μ . Therefore we get

$$((T_\mu - \lambda)u, u) = \int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu g(x)|u|^2 - \lambda|u|^2) \, dx.$$

In the next lemma we will show that the map $(T_\mu - \lambda)$ is coercive.

Lemma 3.2 *For each $\lambda \in (0, \lambda_1(\Omega))$, there exist $\mu(\lambda) > 0$ such that $a_\mu \geq (\lambda + \lambda_1(\Omega))/2$ whenever $\mu \geq \mu(\lambda)$. As a consequence*

$$((T_\mu - \lambda)u, u) \geq \beta_\lambda \|u\|_\mu^2$$

for all $u \in E$, $\mu \geq \mu(\lambda)$, where $\beta_\lambda := (\lambda_1(\Omega) - \lambda)/(\lambda_1(\Omega) + \lambda)$.

Proof. Assume by contradiction that there exist a sequence $\mu_m \rightarrow \infty$ such that $a_{\mu_m} < (\lambda + \lambda_1(\Omega))/2$ for all m and $a_{\mu_m} \rightarrow \theta \leq (\lambda + \lambda_1(\Omega))/2$. Let us consider a minimizing sequence $\{u_m\} \in E$ such that $|u_m|_2 = 1$ and $((T_{\mu_m} - a_{\mu_m})u_m, u_m) \rightarrow 0$. This implies

$$\begin{aligned} \|u_m\|_{\mu_m}^2 &= \int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu_m g(x)|u_m|^2) \, dx \\ &= ((T_{\mu_m} - a_{\mu_m})u_m, u_m) + a_{\mu_m}(u_m, u_m) \\ &\leq ((T_{\mu_m} - a_{\mu_m})u_m, u_m) + (1 + a_{\mu_m})|u_m|_2^2 \\ &\leq (1 + \lambda_1(\Omega)) \end{aligned}$$

for large m , using $\lambda < \lambda_1(\Omega)$ and $\theta \leq (\lambda + \lambda_1(\Omega))/2$. So using Lemma 3.1, we get $u \in H_A^{0,1}(\Omega)$ such that $u_m \rightharpoonup u$ weakly in E and $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$ as $m \rightarrow \infty$. Therefore $|u|_2 = 1$ and $\liminf_{m \rightarrow \infty} |\nabla_A u_m|_2^2 \geq |\nabla_A u|_2^2$. Since $g \geq 0$ and $\mu_m \rightarrow \infty$, we have

$$\begin{aligned} \int_{\Omega} (|\nabla_A u|^2 - \theta|u|^2) &\leq \liminf_{m \rightarrow \infty} \int_{\mathbb{R}^n} (|\nabla_A u_m|^2 + \mu_m g(x)|u_m|^2 - a_{\mu_m}|u_m|^2) \, dx \\ &= \liminf_{m \rightarrow \infty} ((T_{\mu_m} - a_{\mu_m})u_m, u_m) = 0. \end{aligned}$$

Hence

$$\int_{\Omega} |\nabla_A u|^2 \, dx \leq \theta \leq \frac{\lambda + \lambda_1(\Omega)}{2} < \lambda_1(\Omega)$$

which is a contradiction to the definition of $\lambda_1(\Omega)$. For the second part, let $u \in E$ and $\mu \geq \mu(\lambda)$ then $a_\mu \leq \frac{\|u\|_\mu^2}{|u|_2^2}$ which gives

$$\lambda |u|_2^2 \leq \frac{2\lambda \|u\|_\mu^2}{\lambda + \lambda_1(\Omega)}.$$

Therefore

$$((T_\mu - \lambda)u, u) \geq \|u\|_\mu^2 - \lambda |u|_2^2 \geq \beta_\lambda \|u\|_\mu^2,$$

where $\beta_\lambda = \frac{\lambda_1 - \lambda}{\lambda_1 + \lambda}$. ■

Our next lemma assures that all $(PS)_c$ sequences are bounded.

Lemma 3.3 *Let $\{u_m\}$ be a $(PS)_c$ sequence for $I_{\lambda,\mu}$. If $\lambda \in (0, \lambda_1(\Omega))$ and $\mu \geq \mu(\lambda)$, then $\{u_m\}$ is bounded in E and*

$$\lim_{m \rightarrow \infty} ((T_\mu - \lambda)u_m, u_m) = \lim_{m \rightarrow \infty} B(u_m) = \frac{2c(2n - \alpha)}{(n + 2 - \alpha)},$$

where $B(\cdot)$ is defined as in (2.2).

Proof. Using definitions of $I_{\lambda,\mu}$ and T_μ , we get

$$\begin{aligned} I_{\lambda,\mu}(u_m) - \frac{1}{22_\alpha^*} (I'_{\lambda,\mu}(u_m), u_m) &= \left(\frac{2_\alpha^* - 1}{22_\alpha^*} \right) \int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu g(x)|u_m|^2 - \lambda |u_m|^2) \, dx \\ &= \frac{n + 2 - \alpha}{2(2n - \alpha)} ((T_\mu - \lambda)u_m, u_m) \end{aligned} \quad (3.4)$$

and

$$I_{\lambda,\mu}(u_m) - \frac{1}{2} \langle I'_{\lambda,\mu}(u_m), u_m \rangle = \left(\frac{2_\alpha^* - 1}{22_\alpha^*} \right) B(u_m) = \frac{n + 2 - \alpha}{2(2n - \alpha)} B(u_m). \quad (3.5)$$

Using Lemma 3.2 and (3.4), we get

$$\begin{aligned} c - \frac{1}{22_\alpha^*} o_1(\|u_m\|_\mu) &\geq \frac{n + 2 - \alpha}{2(2n - \alpha)} ((T_\mu - \lambda)u_m, u_m) \\ &\geq \beta_\lambda \|u_m\|_\mu^2. \end{aligned}$$

This implies $\{u_m\}$ is bounded in E . Taking limit $m \rightarrow \infty$ in (3.4), we get

$$\lim_{m \rightarrow \infty} \frac{n + 2 - \alpha}{2(2n - \alpha)} = \frac{2c(2n - \alpha)}{n + 2 - \alpha},$$

and taking limit $m \rightarrow \infty$ in (3.5), we get

$$\lim_{m \rightarrow \infty} B(u_m) = \frac{2c(2n - \alpha)}{n + 2 - \alpha}.$$

This completes the proof. ■

Let

$$S_A := \inf_{u \in H_A^1(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu g(x)|u|^2) dx}{B(u)^{\frac{n-2}{2n-\alpha}}} = \inf_{u \in H_A^1(\mathbb{R}^n)} \frac{\|u\|_\mu^2}{B(u)^{\frac{n-2}{2n-\alpha}}}.$$

For the preceding sections, if necessary, we assume $\mu(\lambda) \geq \lambda/M$, where M is defined in (g2). Thus,

$$\mu M - \lambda \geq 0, \text{ for all } \mu \geq \mu(\lambda). \quad (3.6)$$

Before proving our proposition, we recall Lemma 3.5 of [1] here.

Lemma 3.4 *Ψ is a C^1 functional where Ψ and Ψ' map bounded sets to bounded sets. Ψ is weakly sequentially lower semicontinuous and Ψ' is weakly sequentially semicontinuous. If $u_m \rightharpoonup v$ weakly in E , then there exists a subsequence $v_m \rightharpoonup v$ weakly such that*

$$\Psi(u_m) - \Psi(u_m - v_m) \rightarrow \Psi(v) \text{ in } \mathbb{R}$$

$$\Psi'(u_m) - \Psi'(u_m - v_m) \rightarrow \Psi'(v) \text{ in } E^*.$$

Proposition 3.5 *If $\lambda \in (0, \lambda_1(\Omega))$ and $\mu \geq \mu(\lambda)$, then the functional $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition, for all*

$$c \in \left(-\infty, \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}} \right).$$

Proof. Let $\{u_m\} \subset E$ be a sequence such that $I_{\lambda,\mu}(u_m) \rightarrow c < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}$ and $I'_{\lambda,\mu}(u_m) \rightarrow 0$ as $m \rightarrow \infty$. By Lemma 3.3, $\{u_m\}$ is bounded in E that is $\|u_m\|_\mu \leq K_1$ for some constant $K_1 > 0$ and for all m . Therefore, there exist a subsequence (still denoted by $\{u_m\}$) such that $u_m \rightharpoonup u$ weakly in E , $u_m \rightarrow u$ strongly in $L_{\text{loc}}^2(\mathbb{R}^n)$ and $u_m(x) \rightarrow u(x)$ a.e. in $x \in \mathbb{R}^n$. Using Hardy-Littlewood-Sobolev inequality, there exist $K_2 > 0$ such that

$$|B(u_m)| \leq K_2.$$

Similarly, for any $\phi \in E$

$$\langle B'(u), \phi \rangle = \int_{\mathbb{R}^n} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^*-2} u \bar{\phi} dx$$

defines a bounded linear functional. Also, B is weakly lower semicontinuous and weakly sequentially continuous, since $B \in C^1(E, \mathbb{R})$. Therefore using 3.4, we get

$$\langle B'(u_m), \phi \rangle \rightarrow \langle B'(u), \phi \rangle$$

for any $\phi \in E$ when $m \rightarrow \infty$. Thus we get

$$\langle I'_{\lambda,\mu}(u), \phi \rangle = \lim_{m \rightarrow \infty} \langle I'_{\lambda,\mu}(u_m), \phi \rangle = 0$$

showing that u is a weak solution of $(P_{\lambda,\mu})$.

Let $\widetilde{u}_m = u_m - u$, so by Lemma 2.3 of [18] we have

$$B(u_m) - B(\widetilde{u}_m) \rightarrow B(u) \quad (3.7)$$

as $m \rightarrow \infty$. Also since $I'_{\lambda,\mu}(u_m) \rightarrow 0$, we get

$$((T_\mu - \lambda)u_m, u_m) - B(u_m) \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.8)$$

Then using (3.7) and (3.8), we get

$$\lim_{m \rightarrow \infty} ((T_\mu - \lambda)\widetilde{u_m}, \widetilde{u_m}) - B(\widetilde{u_m}) = 0.$$

Let $\lim_{m \rightarrow \infty} ((T_\mu - \lambda)\widetilde{u_m}, \widetilde{u_m}) = \lim_{m \rightarrow \infty} B(\widetilde{u_m}) = b$ (say). It is easy to show that $I_{\lambda,\mu}(u) > 0$ and using this

$$\frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}} > c = \lim_{m \rightarrow \infty} I_{\lambda,\mu}(u_m) \geq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla_A \widetilde{u_m}|^2 dx - \frac{1}{22_\alpha^*} B(\widetilde{u_m}) + o_m(1).$$

This implies

$$b \leq \frac{2c(2n-\alpha)}{n+2-\alpha} < S_A^{\frac{2n-\alpha}{n+2-\alpha}}. \quad (3.9)$$

Let $D = \{x \in \mathbb{R}^n : g(x) \leq M\}$, M is defined as (g2). Then using similar arguments as in Lemma 3.1, we can show that

$$\int_D |\widetilde{u_m}|^2 dx \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Then using (3.6), definition S_A and above equation

$$\begin{aligned} S_A B(\widetilde{u_m})^{\frac{1}{2_\alpha^*}} &\leq \int_{\mathbb{R}^n} |\nabla_A \widetilde{u_m}|^2 dx + \mu \int_{\mathbb{R}^n} g(x) |\widetilde{u_m}|^2 dx \\ &\leq \int_{\mathbb{R}^n} |\nabla_A \widetilde{u_m}|^2 dx + \int_{\mathbb{R}^n \setminus D} (\mu g(x) - \lambda) |\widetilde{u_m}|^2 dx + \int_D (\lambda + g(x)) |\widetilde{u_m}|^2 dx \\ &\leq ((T_\mu - \lambda)\widetilde{u_m}, \widetilde{u_m}) + (\lambda + M) \int_D |\widetilde{u_m}|^2 dx \\ &= ((T_\mu - \lambda)\widetilde{u_m}, \widetilde{u_m}) + o_m(1). \end{aligned}$$

Passing on the limits we get $b \geq S_A^{\frac{2n-\alpha}{n+2-\alpha}}$ which is a contradiction to (3.9). Therefore, $b = 0$ that is $u_m \rightarrow u$ strongly in E as $m \rightarrow \infty$. \blacksquare

4 Proof of main Theorems

Before proving the main theorems, we prove the results below that will help to achieve our goal. The theorem below is similar to Theorem 1.1 of [6].

Theorem 4.1 *If $g \geq 0$, $g \in L_{loc}^{n/2}(\mathbb{R}^n)$ and $A \in L_{loc}^n(\mathbb{R}^n, \mathbb{R}^n)$, then the infimum S_A is attained if and only if $g \equiv 0$ and $\text{curl } A \equiv 0$.*

Proof. At first, we prove that $S_A = S_{H,L}$. By Proposition 2.2 and Theorem 2.1, we have

$$S_{H,L} \leq \frac{\int_{\mathbb{R}^n} (|\nabla|u||^2 + g(x)|u|^2) \, dx}{B(u)^{\frac{n-2}{2n-\alpha}}} \leq \frac{\int_{\mathbb{R}^n} (|\nabla_A u|^2 + g(x)|u|^2) \, dx}{B(u)^{\frac{n-2}{2n-\alpha}}}.$$

This implies $S_{H,L} \leq S_A$. Without loss of generality, we assume $0 \in \Omega$ and $B_\delta \subset \Omega \subset B_{2\delta}(B_r)$ is an open ball of radius r and center 0 . Let

$$U_\epsilon(x) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{\epsilon}{\epsilon^2 + |x|^2} \right)^{\frac{n-2}{4}}$$

and $u_\epsilon(x) = \psi(x)U_\epsilon(x)$, where $\psi \in C_c^\infty(\mathbb{R}^n, [0, 1])$ such that $\psi \equiv 1$ in $B(0, \delta)$ and $\psi \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. We recall the following asymptotic estimates from [18]

$$(1) \quad |\nabla u_\epsilon|_2^2 = C(n, \alpha)^{\frac{n(n-2)}{2(2n-\alpha)}} S_{H,L}^{\frac{n}{2}} + o(\epsilon^{n-2}).$$

(2)

$$|u_\epsilon|_2^2 = \begin{cases} d\epsilon^2 |\ln \epsilon| + o(\epsilon^2) & \text{if } n = 4 \\ d\epsilon^2 + o(\epsilon^{n-2}) & \text{if } n \geq 5, \end{cases}$$

where d is a positive constant.

$$(3) \quad B(u_\epsilon)^{(n-2)/(2n-\alpha)} \geq \left(C(n, \alpha)^{n/2} S_{H,L}^{(2n-\alpha)/2} - o(\epsilon^{n-\alpha/2}) \right)^{(n-2)/(2n-\alpha)}.$$

$$(4) \quad |u_\epsilon|_{2^*}^{2^*} = S^{n/2} + o(\epsilon^n), \text{ where } S \text{ is the best Sobolev constant and } 2^* = 2n/(n-2).$$

Also up to a subsequence $u_\epsilon(x) \rightarrow 0$ a.e. in \mathbb{R}^n as $\epsilon \rightarrow 0$. Then

$$\int_{\mathbb{R}^n} g(x)|u_\epsilon|^2 \, dx = \langle g, |u_\epsilon|^2 \rangle \rightarrow 0 \text{ as } \epsilon \rightarrow 0,$$

where the duality product is taken with respect to $L^{n/2}(\mathbb{R}^n)$ and $L^{2^*/2}(\mathbb{R}^n)$. Since $A \in L_{\text{loc}}^n(\mathbb{R}^n, \mathbb{R}^n)$, using similar argument we have

$$\int_{\mathbb{R}^n} |Au_\epsilon|^2 \, dx = \langle |A|^2, |u_\epsilon|^2 \rangle \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Now, if we choose $\epsilon > 0$ small enough then using (1) and (3) we have

$$\begin{aligned} \frac{\int_{\mathbb{R}^n} (|\nabla_A u_\epsilon|^2 + g(x)|u_\epsilon|^2) \, dx}{B(u_\epsilon)^{\frac{n-2}{2n-\alpha}}} &= \frac{\int_{\mathbb{R}^n} (|\nabla u_\epsilon|^2 + |Au_\epsilon|^2 + g(x)|u_\epsilon|^2) \, dx}{B(u_\epsilon)^{\frac{n-2}{2n-\alpha}}} \\ &\leq \frac{C(n, \alpha)^{\frac{n(n-2)}{2(2n-\alpha)}} S_{H,L}^{n/2} + 2o(\epsilon)}{\left(C(n, \alpha)^{n/2} S_{H,L}^{(2n-\alpha)/2} - o(\epsilon^{n-\alpha/2}) \right)^{(n-2)/(2n-\alpha)}} \\ &\leq S_{H,L} + \delta, \end{aligned}$$

where δ is a positive constant. This implies $S_A \leq S_{H,L}$, therefore $S_A = S_{H,L}$. Let u be minimizer of S_A normalized by $B(u) = 1$. Then

$$S_A = \int_{\mathbb{R}^n} (|\nabla_A u|^2 + \mu g(x)|u|^2) \, dx \geq \int_{\mathbb{R}^n} |\nabla_A u|^2 \geq \int_{\mathbb{R}^n} |\nabla|u||^2 \, dx \geq S_{H,L}. \quad (4.1)$$

Consequently, $|u(x)| = U_\epsilon(x - a)/B(U_\epsilon)$, for some $a \in \mathbb{R}^n$ because the minimizers of $S_{H,L}$ are of the form U_ϵ which are invariant under translation and dilation (Lemma 1.2, 1.3 of [18]). We can take $|u| > 0$ and the equality in (4.1) occurs when $\mu g(x) = 0$. Moreover, due to this reason, the diamagnetic inequality in Theorem 2.1 must have equality a.e. Therefore $\text{Im}((\nabla_A u)\bar{u}/\|u\|) = 0$ that is $A = -\text{Im}(\nabla u/u)$. This implies $\text{curl}(\nabla u/u) = 0$. The condition is also sufficient, the proof follows similarly as in Theorem 1.1 of [6]. ■

The next step to prove our main theorem is introducing the Nehari manifold. Let

$$\mathcal{N}_{\lambda,\mu} = \{u \in E \setminus \{0\} : \langle I'_{\lambda,\mu}(u), u \rangle = 0\} = \{u \in E \setminus \{0\} : ((T_\mu - \lambda)u, u) = B(u)\}.$$

The critical points of $I_{\lambda,\mu}$ lie in $\mathcal{N}_{\lambda,\mu}$. Let $X = \{v \in E : B(v) = 1\}$. Using fibering map analysis, we say that for each $u \in E$, there exist

$$t_u = \left(\frac{((T_\mu - \lambda)(u), u)}{B(u)} \right)^{\frac{n-2}{2(n+2-\alpha)}}$$

such that $t_u u \in \mathcal{N}_{\lambda,\mu}$. Using Proposition 1.1 of [28], we get $\mathcal{N}_{\lambda,\mu}$ is radially diffeomorphic to X via the map

$$u \mapsto \left(\frac{((T_\mu - \lambda)(u), u)}{B(u)} \right)^{\frac{n-2}{2(n+2-\alpha)}} u.$$

On $\mathcal{N}_{\lambda,\mu}$,

$$I_{\lambda,\mu}(u) = \frac{n+2-\alpha}{2(2n-\alpha)} ((T_\mu - \lambda)(u), u),$$

so we get

$$k_{\lambda,\mu} := \inf_{u \in \mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u) = \frac{n+2-\alpha}{2(2n-\alpha)} \inf_{v \in X} ((T_\mu - \lambda)(v), v)^{\frac{2n-\alpha}{n+2-\alpha}}.$$

Proposition 4.2 *Let $u \in \mathcal{N}_{\lambda,\mu}$ be a critical point of $I_{\lambda,\mu}$ such that $I_{\lambda,\mu}(u) < 2k_{\lambda,\mu}$. Then u does not change sign that is $|u|$ is a solution of $(P_{\lambda,\mu})$.*

Proof. Since u is a critical point of $I_{\lambda,\mu}$, for every $w \in E$

$$((T_\mu - \lambda)(u), w) = \text{Re} \left(\int_{\mathbb{R}^n} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^*} u \bar{w} \, dx \right).$$

Let $w = u^\pm$ in above equation, where $u^\pm = \pm \max\{\pm u, 0\}$. Suppose u changes sign, so u^+ and u^- are both nonzero and $u^\pm \in \mathcal{N}_{\lambda,\mu}$. Then $I_{\lambda,\mu}(u) = I_{\lambda,\mu}(u^+) + I_{\lambda,\mu}(u^-) \geq 2k_{\lambda,\mu}$ which is a contradiction. Thus, u does not change sign and we have the conclusion. ■

Now consider any domain $\mathcal{Q} \subset \mathbb{R}^n$. As we defined $I_{\lambda,\mu}$, in a similar manner we define

$$\begin{aligned} I_{\mu,\mathcal{Q}}(u) &= \frac{1}{2} \int_{\mathcal{Q}} (|\nabla_A u|^2 + \lambda |u|^2) \, dx - \frac{1}{22_\alpha^*} \int_{\mathcal{Q}} \int_{\mathcal{Q}} \frac{|u(x)|^{2_\alpha^*} |u(y)|^{2_\alpha^*}}{|x-y|^\alpha} \, dx dy \\ &= \frac{1}{2} ((T_0 - \lambda)(u), u) - \frac{1}{22_\alpha^*} B(u) \end{aligned}$$

for $u \in H_A^{0,1}(\mathcal{Q})$. This is an energy functional associated to the problem

$$(P_\lambda) \begin{cases} (-i\nabla + A(x))^2 u = \lambda u + (|x|^\alpha * |u|^{2_\alpha^*}) u^{2_\alpha^*-1}, & u > 0 \quad \text{in } \mathcal{Q} \\ u = 0 & \text{on } \partial\mathcal{Q}. \end{cases}$$

The Nehari manifold associated to (P_λ) is given by

$$\mathcal{N}_{\lambda,\mathcal{Q}} = \left\{ u \in H_A^{0,1}(\mathcal{Q}) \setminus \{0\} : ((T_0 - \lambda)(u), u) = B(u) \right\}$$

which is radially diffeomorphic to $X_\mathcal{Q} = \{v \in H_A^{0,1}(\mathcal{Q}) : B(v) = 1\}$. We set

$$k_{\mu,\mathcal{Q}} := \inf_{u \in \mathcal{N}_{\lambda,\mathcal{Q}}} I_{\lambda,\mathcal{Q}}(u) = \frac{n+2-\alpha}{2(2n-\alpha)} \inf_{v \in X_\mathcal{Q}} ((T_0 - \lambda)(u), u)^{\frac{2n-\alpha}{n+2-\alpha}}.$$

Lemma 4.3 *If $\lambda \in (0, \lambda_1(\Omega))$ and $\mu \geq \mu(\lambda)$, then the following holds:*

$$\frac{n+2-\alpha}{2(2n-\alpha)} (\beta_\lambda S_A)^{\frac{2n-\alpha}{n+2-\alpha}} \leq k_{\lambda,\mu} \leq k_{\lambda,\Omega} < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}.$$

Proof. Using Lemma 3.2, we have $\beta_\lambda \|v\|_A^2 \leq \beta_\lambda \|v\|_\mu^2 \leq ((T_\mu - \lambda)(u), u)$. This implies, taking infimum over X , we get

$$\frac{n+2-\alpha}{2(2n-\alpha)} (\beta_\lambda S_A)^{\frac{2n-\alpha}{n+2-\alpha}} \leq k_{\lambda,\mu}.$$

This gives the first inequality. Now, for the second inequality, since $X_\Omega \subset X$ we get $k_{\lambda,\mu} \leq k_{\lambda,\Omega}$. We aim to show that

$$k_{\lambda,\Omega} < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}.$$

Let U_ϵ and u_ϵ be as defined in Lemma 4.1. Define

$$J_\lambda(u) = \frac{\int_{\mathbb{R}^n} (|\nabla_A u|^2 - \lambda |u|^2) \, dx}{B(u)^{\frac{n-2}{2n-\alpha}}}.$$

Let A be continuous at $'0'$ and $\gamma(x) := -\sum A_j(0)x_j$. Then it is easy to check that $(A + \nabla\gamma)(0) = 0$ and therefore by continuity of A at $'0'$ we get $\delta > 0$ such that

$$|(A + \nabla\gamma)(x)|^2 \leq \tilde{k} < \lambda, \text{ for all } |x| < \delta.$$

Also let $v_\epsilon(x) = \psi(x)U_\epsilon(x) \exp(i\gamma(x))$, where

$$\psi(x) = \begin{cases} 1 & \text{in } B(0, \delta/2) \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}.$$

Using (1) of Lemma 4.1, we get

$$\begin{aligned} \int_{\mathbb{R}^n} (|\nabla_A v_\epsilon|^2 - \lambda |v_\epsilon|^2) \, dx &= \int_{\mathbb{R}^n} (|(-i\nabla + A)(\psi U_\epsilon \exp(i\gamma))|^2 - \lambda \psi^2 U_\epsilon^2) \, dx \\ &= \int_{\mathbb{R}^n} (|\nabla(\psi U_\epsilon)|^2 + \psi^2 U_\epsilon^2 |\nabla\gamma + A|^2 - \lambda \psi^2 U_\epsilon^2) \, dx \\ &\leq C(n, \alpha)^{\frac{n(n-2)}{2(2n-\alpha)}} S_{H,L}^{\frac{n}{2}} + o(\epsilon^{n-2}) + (\tilde{k} - \lambda) \int_{B(0, \frac{\delta}{2})} U_\epsilon^2 \, dx. \end{aligned}$$

Moreover, using (3) of Lemma 4.1, we get

$$B(v_\epsilon) = B(u_\epsilon) \geq \left(C(n, \alpha)^{n/2} S_{H,L}^{(2n-\alpha)/2} - o(\epsilon^{n-\alpha/2}) \right)^{(n-2)/(2n-\alpha)}.$$

It is a standard result that for $\epsilon > 0$ small enough

$$\int_{B(0, \frac{\epsilon}{2})} U_\epsilon^2 dx = \begin{cases} C\epsilon^2 |\log \epsilon| & \text{if } n = 4 \\ C\epsilon^2 & \text{if } n \geq 5, \end{cases}$$

where $C > 0$ is a constant depending only on n . Therefore since $\tilde{k} - \lambda < 0$, when $n \geq 5$ we get

$$J_\lambda(v_\epsilon) \leq \frac{C(n, \alpha)^{\frac{n(n-2)}{2(2n-\alpha)}} S_{H,L}^{\frac{n}{2}} + o(\epsilon^{n-2}) + (\tilde{k} - \lambda)C\epsilon^2}{\left(C(n, \alpha)^{n/2} S_{H,L}^{(2n-\alpha)/2} - o(\epsilon^{n-\alpha/2}) \right)^{(n-2)/(2n-\alpha)}} < S_{H,L} = S_A.$$

This implies

$$k_{\lambda, \Omega} \leq \frac{n+2-\alpha}{2(2n-\alpha)} J_\lambda(v_\epsilon)^{\frac{2n-\alpha}{n+2-\alpha}} < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}$$

that is the last inequality. Similarly the result follows for $n = 4$. ■

Proof of Theorem 1.3: Let $\{u_m\}$ be a minimizing sequence for $I_{\lambda, \mu}$ on $\mathcal{N}_{\lambda, \mu}$. Then by Ekeland Variational Principle [17], $\{u_m\}$ becomes a Palais-Smale sequence. Using Proposition 3.5 and Lemma 4.3, we conclude that there exist a subsequence of $\{u_m\}$ that converges to least energy solution, say u_μ of $(P_{\lambda, \mu})$. ■

Proof of Theorem 1.4: Let $\{u_m\}$ be a sequence of solution for the problem (P_{λ, μ_m}) such that $\lambda \in (0, \lambda_1(\Omega))$, $\mu_m \rightarrow \infty$ and

$$I_{\lambda, \mu_m}(u_m) = ((T_{\mu_m} - \lambda)(u_m), u_m) \rightarrow c < \frac{n+2-\alpha}{2(2n-\alpha)} S_{H,L}^{\frac{2n-\alpha}{n+2-\alpha}}.$$

By Lemma 3.2, $((T_{\mu_m} - \lambda)u_m, u_m) \geq \beta_\lambda \|u_m\|_{\mu_m}^2$ which implies $\beta_\lambda \|u_m\|_{\mu_m}^2 \leq c + o_m(1)$. So, $\{u_m\}$ is bounded in E . So using Lemma 3.1, we say that there exist $u \in H_A^{0,1}(\Omega)$ such that, upto a subsequence, $u_m \rightharpoonup u$ weakly in E and $u_m \rightarrow u$ strongly in $L^2(\mathbb{R}^n)$. Since u_{μ_m} solves (P_{λ, μ_m}) ,

$$\operatorname{Re} \left(\int_{\mathbb{R}^n} \nabla_A u_m \overline{\nabla_A v} dx + \int_{\mathbb{R}^n} (\mu_m g(x) - \lambda) u_m \bar{v} - (|x|^{-\alpha} * |u_m|^{2_\alpha^*}) |u_m|^{2_\alpha^*-2} u_m \bar{v} dx \right) = 0$$

for every $v \in E$. Since $\Omega = \{x \in \mathbb{R}^n : g(x) = 0\}$, for any $v \in H_A^{0,1}(\Omega)$, $\mu_m \int_{\mathbb{R}^n} g(x) u_m \bar{v} dx = 0$. Letting $n \rightarrow \infty$, we obtain

$$\operatorname{Re} \left(\int_{\mathbb{R}^n} \nabla_A u \overline{\nabla_A v} dx + \int_{\mathbb{R}^n} \lambda u \bar{v} dx - \int_{\mathbb{R}^n} (|x|^{-\alpha} * |u|^{2_\alpha^*}) |u|^{2_\alpha^*-2} u \bar{v} dx \right) = 0$$

which implies u is a solution of (P_λ) . Let $\tilde{u}_m = u_m - u$, then $\tilde{u}_m \rightharpoonup 0$ weakly in E and $\tilde{u}_m \rightarrow 0$ strongly in $L^2(\mathbb{R}^n)$. Therefore,

$$((T_{\mu_m} - \lambda)\tilde{u}_m, \tilde{u}_m) = ((T_{\mu_m} - \lambda)\tilde{u}_m, \tilde{u}_m) + ((T_0 - \lambda)u, u) + o(1). \quad (4.2)$$

By Lemma 2.3 of [18], we get

$$B(u_m) - B(\tilde{u}_m) \rightarrow B(u) \text{ as } m \rightarrow \infty.$$

So it is easy to see that

$$((T_{\mu_m} - \lambda)\tilde{u}_m, \tilde{u}_m) - B(\tilde{u}_m) = o(1). \quad (4.3)$$

We claim that $B(\tilde{u}_m) \rightarrow 0$ as $m \rightarrow \infty$. Suppose not, that is $B(\tilde{u}_m) \rightarrow b > 0$ as $m \rightarrow \infty$. Using (4.3) and arguments as in Lemma 3.5, we get

$$S_A B(\tilde{u}_m)^{\frac{n-2}{2n-\alpha}} \leq \int_{\mathbb{R}^n} |\nabla_A \tilde{u}_m|^2 \, dx \leq ((T_{\mu_m} - \lambda)\tilde{u}_m, \tilde{u}_m) + o(1) = B(\tilde{u}_m) + o(1).$$

Consequently, we get that $S_A \leq B(\tilde{u}_m)^{\frac{n+2-\alpha}{2n-\alpha}} + o(1)$ which implies

$$S_A^{\frac{2n-\alpha}{n+2-\alpha}} \leq \lim_{m \rightarrow \infty} B(u_m). \quad (4.4)$$

It is not hard to find that $\lim_{m \rightarrow \infty} B(u_m) = \frac{2c(2n-\alpha)}{(n+2-\alpha)}$. From (4.4)

$$S_A^{\frac{2n-\alpha}{n+2-\alpha}} \leq \lim_{m \rightarrow \infty} B(u_m) < S_A^{\frac{2n-\alpha}{n+2-\alpha}}$$

which is a contradiction. Hence, $\lim_{m \rightarrow \infty} B(u_m) = 0$ and $((T_{\mu_m} - \lambda)\tilde{u}_m, \tilde{u}_m) \rightarrow 0$ as $m \rightarrow \infty$. Using (4.2), we get

$$((T_0 - \lambda)u, u) = \lim_{m \rightarrow \infty} ((T_{\mu_m} - \lambda)u_m, u_m). \quad (4.5)$$

As we proved in Lemma 3.1, here also we can show that $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, so $u_m = \tilde{u}_m$ in $\mathbb{R}^n \setminus \Omega$. Since $g \equiv 0$ in Ω ,

$$\int_{\mathbb{R}^n} g(x)u_m^2 \, dx \leq \mu_m \int_{\Omega} g(x)\tilde{u}_m^2 \, dx \leq ((T_{\mu_m} - \lambda)(\tilde{u}_m), \tilde{u}_m) + o(1).$$

Therefore, $\int_{\mathbb{R}^n} g(x)u_m^2 \, dx = 0$ and (4.5) implies

$$\lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} |\nabla_A u_m|^2 \, dx = \int_{\mathbb{R}^n} |\nabla_A u|^2 \, dx$$

that is $u_m \rightarrow u$ in E . ■

Corollary 4.4 *If $\lambda \in (0, \lambda_1(\Omega))$, then $\lim_{\mu \rightarrow \infty} k_{\lambda, \mu} = k_{\lambda, \Omega}$.*

Proof. By Lemma 4.3, $k_{\lambda, \mu} \rightarrow c \leq k_{\lambda, \Omega} < \frac{n+2-\alpha}{2(2n-\alpha)} S_A^{\frac{2n-\alpha}{n+2-\alpha}}$. Theorem 1.3 implies that $k_{\lambda, \mu}$ is achieved for $\mu \geq \mu(\lambda)$. Therefore, Theorem 1.4 says c must be achieved by $I_{\mu, \Omega}$ on $N_{\lambda, \Omega}$. Hence $c \geq k_{\mu, \Omega}$. ■

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